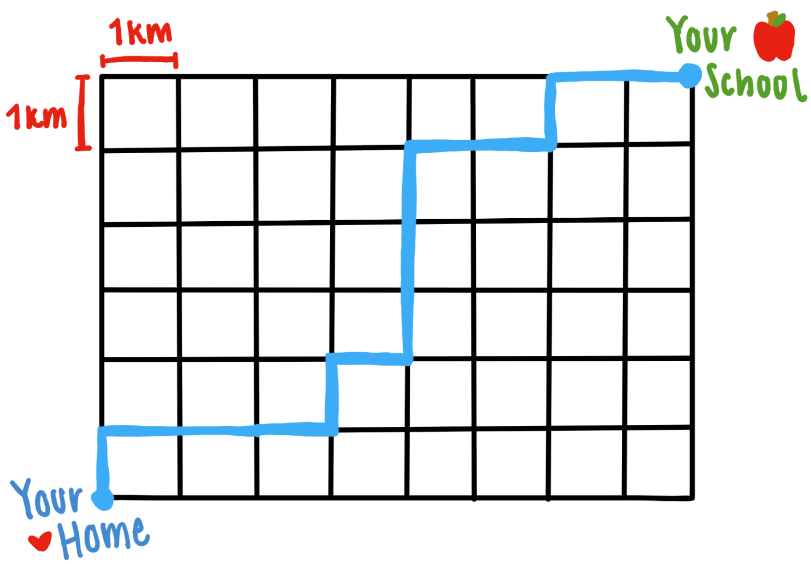
Grade 6 Math Circles
March 4th-8th, 2024
The Binomial Coefficient

Section 1: How to Count

Imagine drawing out a grid where the bottom left corner is your home and where the top right corner is your school, like the drawing below. What if you wanted to figure out how many paths there are from your home to school while only making forward progress toward your destination, how could we possibly count all these paths without having to draw out every single path which exists.



In mathematics the study of counting objects (like paths) is called enumeration, and today we are going to learn some of the fundamental properties which help mathematicians enumerate!

> **Stop and Think**
> What are some ways we could count the number of paths from the home to the school in the drawing above?

Section 2: Factorials

To begin building a foundation which will help us with our goal of being able to count paths we will first introduce the idea of the factorial!

Definition 1

We define the factorial of a whole number n as the product of all the whole numbers from 1 to n, we denote the factorial of a whole number n as $n!$ and say n factorial.

Exercise 2.1

Calculate the following factorials:

1. 2!
2. 3!
3. 0!
4. 5!
5. 6!
6. 20!

Exercise 2.1 Solution

1. $2! = 2$
2. $3! = 6$
3. $0! = 1$ (we'll see why this is true later on)
4. $5! = 120$
5. $6! = 720$
6. $20! = 2432902008176640000$

Division with Factorials

A common practice when working with factorials is to know how to manipulate them through multiplication or division. For what we are working on today division will be much more useful. Let's first start by reviewing what it means to and how to reduce a fraction to lowest terms.

Definition 2

To **reduce** a fraction to **lowest terms** we identify all the *common divisors* of the numerator and denominator and divide both the numerator and denominator by all their common divisors.

Let's do an example together to see what is actually happening when we reduce a fraction.

Example 2.1

Reduce the fraction 15/25 to lowest terms.

Solution: We first begin by figuring out what the greatest common divisor of 15 and 25 is. The divisors of 15 are 1, 3, 5, and 15, the divisors of 25 are 1, 5, and 25, therefore the greatest common divisor of 15 and 25 is 5.
Now we will divide both 15 and 25 by 5 to give us 3 and 5. Thus, the fraction 15/25 reduced to lowest terms is 3/5.

Exercise 2.2

Reduce the fraction 75/230 to lowest terms.

Exercise 2.2 Solution

The divisors of 75 are 1, 3, 5, 15, 25, and 75 . The divisors of 230 are 1, 2, 5, 10, 23, 46, 115, and 230. Therefore the greatest common divisor of 75 and 230 is 5. Now we will divide both 75 and 230 by 5 to give us 15 and 46. Thus, the fraction 75/230 reduced to lowest terms is 15/46.

Now that we understand what it looks like to reduce a fraction to lowest terms, what would it look like to reduce a fraction where the numerator and denominator are both factorials into lowest terms? This is actually a lot easier than it seems, lets look at an example together!!

Example 2.2

Reduce the fraction $7!/9!$ to lowest terms.

Solution:

Normally we start by figuring out what the greatest common divisor of 7! and 9! is, but instead lets start by rewriting 7! and 9! using the definition of what a factorial is.

Recall- the factorial of a whole number n as the product of all the whole numbers from 1 to n. so we have the following:

$7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$ and $9! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$ so we can rewrite the fraction $\dfrac{7!}{9!}$ as $\dfrac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}$.

With this we can see that we can cancel out $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$ from the numerator and denominator to get $\dfrac{7!}{9!} = \dfrac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9} = \dfrac{1}{8 \times 9}$.

Thus in lowest terms $7!/9! = 1/72$

Exercise 2.3

Reduce the fraction $9!/3!$ to lowest terms.

Exercise 2.3 Solution

$9! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$ and $3! = 1 \times 2 \times 3$ so we can rewrite the fraction $9!/3!$ as $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3}$. Now from the numerator and denominator we can cancel out the $1 \times 2 \times 3$ to get $\frac{4 \times 5 \times 6 \times 7 \times 8 \times 9}{1} = 4 \times 5 \times 6 \times 7 \times 8 \times 9 = 60480$

Section 3: The Binomial Coefficient

Now that we have understood what it looks like to simplify fractions of factorials, lets return to our original goal of learning the fundamentals of enumeration!

Activity: Balls in a Box

Consider a box containing 9 distinct balls, we want to figure out all the different pairs of balls we could pick from this box, that is we want to know in how many ways we can **choose 2** balls from the **9** balls we have available.

When we are picking out the first ball from the box there are a total of 9 balls we could pick from. Once that ball is selected we cannot choose it as an option again!
So now when we are picking out the second ball from the box there are a total of 8 balls we could pick from.

Thus we have 9 options for our first selection and 8 options for our second selection, which means we have $9 \times 8 = 72$ total options for our final selection.

But we are not done yet because we are actually double counting the pairs of balls which we select from our box. Let B_1 be the ball we choose on our first selection and let B_2 be the ball we choose on our second selection, the resulting pair of balls we end up with is B_1 and B_2. Now what if we reversed their order so that B_2 is the ball we choose on our first selection and B_1 is the ball we choose on our second selection, then the resulting pair of balls we end up with is still B_1 and B_2!!

This means that the order in which we choose our balls does not matter, so we have to divide our 72 total options for our final selection by 2 (which is the number of ways we can order two distinct balls) to make sure that we are not double counting the pairs of balls which we select from our box. This tells us that there are $\frac{9\times 8}{2} = \frac{72}{2} = 36$ total options for our final selection. Therefore there are 36 ways to choose 2 balls from a box containing 9 balls.

Example 3.1

In how many ways can we **choose 3** balls from a box of **9** balls?

Solution:

Using our same logic as in the above activity we have the following;

When we are picking out the first ball from the box there are a total of 9 balls we could pick from. Once that ball is selected we cannot choose it as an option again! So now when we are picking out the second ball from the box there are a total of 8 balls we could pick from. Finally when we are selecting the third and final ball there are a total of 7 balls we could pick from.

Thus we have 9 options for our first selection and 8 options for our second selection, and 7 options for our third selection, so we have $9 \times 8 \times 7 = 504$ total options for our final selection.

But once again we need to realize that the order in which we select these three balls does not matter, and so we have to divide 504 by the number of ways in which we can order 3 distinct balls.

In enumeration (and mathematics in general) we define the number of ways we order n distinct elements is equal to $n!$ so the number of ways in which we can order 3 distinct balls is equal to $3! = 6$.

This tells us that there are $\dfrac{9 \times 8 \times 7}{3!} = \dfrac{504}{6} = 84$ total options for our final selection. Therefore there are 84 ways to choose 3 balls from a box containing 9 balls.

Slowly but surely through these two examples we can see a pattern emerging! When we want to pick k distinct elements from a set containing n total elements we start by taking the decreasing product of k elements starting from n. That is, to pick the first element we have n options, then to pick the second we have $n-1$ options, to pick the third element we have $n-2$ options, and so on!
Then the last step is to divide the product $n \times (n-1) \times (n-2) \times ...$ by the number of ways in which we can order k distinct elements which we have learned is equal to $k!$.

What we want to be able to do is to write a formula which gives us the number of ways to choose k distinct elements from a set containing n total elements. So let's look for a way to figure out how to rewrite the product $n \times (n-1) \times (n-2) \times ...$ in terms of n and k.

Example 3.2

Rewrite the product $9 \times 8 \times 7$ as a fraction of factorials.

Solution: Recall from Example 3.1 that the product $9 \times 8 \times 7$ was the numerator of our expression, since when we are choosing 3 balls from a box of 9 balls we have 9 options for our first selection and 8 options for our second selection, and 7 options for our third selection.

In this example we want to be able to rewrite the product $9 \times 8 \times 7$ as a fraction of factorials. Earlier in the lesson we saw that we can be left with a product like this by cancelling out the remaining terms of a factorial. Since this product's largest number is 9 we know that our fraction is going to have $9!$ in the numerator.

Now it remains to figure out what our denominator is if $9! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$ is our numerator then that means we cancelled out the product $1 \times 2 \times 3 \times 4 \times 5 \times 6$ from our numerator and denominator to end up with $9 \times 8 \times 7$. But notice $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 6!$ so we can write $9 \times 8 \times 7 = 9!/6!$.

Exercise 3.1

Rewrite the product 9×8 as a fraction of factorials.

Exercise 3.1 Solution

In this example we want to be able to rewrite the product 9×8 as a fraction of factorials. Earlier in the lesson we saw that we can be left with a product like this by cancelling out the remaining terms of a factorial. Since this product's largest number is 9 we know that our fraction is going to have $9!$ in the numerator.

Now it remains to figure out what our denominator is if $9! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$ is our numerator then that means we cancelled out the product $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$ from our numerator and denominator to end up with 9×8. But notice $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 7!$ so we can write $9 \times 8 = 9!/7!$.

In both Example 3.2 and Exercise 3.1 we saw a way to rewrite products $9 \times 8 \times 7$ and 9×8 in terms of factorials. But $9 \times 8 \times 7$ was the numerator of our expression, when we are choosing 3 balls from a box of 9 balls (since we have 9 options for our first selection and 8 options for our second selection, and 7 options for our third selection) and 9×8 was the numerator of our expression, when we are choosing 2 balls from a box of 9 balls (since we have 9 options for our first selection and 8 options for our second selection).

So we have that $9 \times 8 \times 7 = 9!/6!$ and note $6 = 9 - 3$ so we can write $9!/6! = 9!/(9-3)!$ and similarly $9 \times 8 = 9!/7!$ and note $7 = 9 - 2$ so we can write $9!/7! = 9!/(9-2)!$.

So lets see how we can use this new information to redo Example 3.1 in a cleaner way!

Example 3.3

In how many ways can we **choose 3** balls from a box **9** balls?

Solution:

When we are picking out the first ball from the box there are a total of 9 balls we could pick from. Once that ball is selected we can not choose it as an option again! So now when we are picking out the second ball from the box there are a total of 8 balls we could pick from. Finally when we are selecting the third and final ball there are a total of 7 balls we could pick from.

Thus we have 9 options for our first selection and 8 options for our second selection, and 7 options for our third selection, so we have $9 \times 8 \times 7 = 9!/6! = 9!/(9-3)!$ total options for our final selection.

But once again we need to realize that the order in which we select these three balls does not matter, and so we have to divide 504 by the number of ways in which we can order 3 distinct balls.

In enumeration (and mathematics in general) we define the number of ways we order n distinct elements is equal to $n!$ so the number of ways in which we can order 3 distinct balls is equal to $3! = 6$.

This tells us that there are $\dfrac{9 \times 8 \times 7}{3!} = \dfrac{9!}{(9-3)! \times 1/3!} = 84$ total options for our final selection. Therefore there are 84 ways to choose 3 balls from a box containing 9 balls.

Thus we've found a way to write out a formula which tells us immediately how many ways k distinct elements can be chosen from a set containing n total elements!

Definition 3

The Binomial Coefficient is a number which tells us in how many ways k distinct elements can be chosen from a set containing n total elements, we write this as $\binom{n}{k} = \frac{n!}{(n-k)! \times k!}$, we read the binomial coefficient as "n choose k".

Note: if $k > n$ then $\binom{n}{k} = 0$.

Exercise 3.2

Calculate the following binomial coefficients;

1. $\binom{4}{3}$
2. $\binom{3}{4}$
3. $\binom{9}{5}$
4. $\binom{9}{7}$

Exercise 3.2 Solution

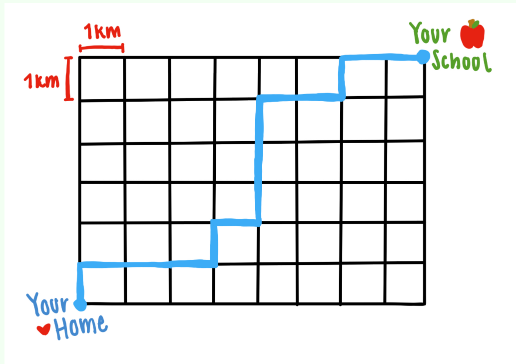
1. 4
2. 0
3. 126
4. 36

Section 4: Lattice Paths

Recall the problem we introduced in the beginning of class, which asked us how many paths are there from home to school which only take steps right and up. Now that we understand the binomial coefficient and how to use it we will find a way to use it in order to solve this problem.

Example 4.1

How many paths from home to school are there in the grid below, which take only take steps right and up?



Solution: The size of the grid is 8 km long and 6 km high and we reach an intersection every kilometer. This tells us that to reach the school we need to travel 8 squares right and 6 squares up, because any other combination of steps would take us to a different point on the grid. Therefore we know we need to travel a total of $8 + 6 = 14$ squares, but we know that we have to **choose** 8 of these 14 squares need to be moved in the right direction. therefore the number of paths from home to school which take only take steps right and up is given by $\binom{14}{8} = 3003$.

Stop and Think

Does it matter that we chose 8 of the 14 squares to be right steps and calculated $\binom{14}{8} = 3003$, could we have instead chosen 6 of the 14 squares to be up steps and calculated $\binom{14}{6}$?

Definition 4

A Lattice Path is a path composed of connected horizontal and vertical line segments (right-steps and up-steps) which pass through a grid. The set of paths of length $a+b$ (a right-steps + b up-steps) is denoted by $\mathcal{L}(a,b)$. The number of paths of length $a+b$ is given by $|\mathcal{L}(a,b)| = \binom{a+b}{b}$

Exercise 4.1 Solution

1. $\binom{6}{2} = 15$
2. $\binom{9}{3} = 84$
3. $\binom{6}{4} = 15$
4. $\binom{14}{6} = 3003$
5. $\binom{11}{2} = 55$

Section 5: Properties of the Binomial Coefficient

So far we've seen two interpretations of the binomial coefficient, one through choosing elements from a set of a fixed size, and the second through lattice paths. Let's now use these interpretations to understand and prove two properties of the binomial coefficient!

Property 1 of the Binomial Coefficient

$\binom{n}{k} = \binom{n}{n-k}$

Example 5.1

Show that $\binom{n}{k} = \binom{n}{n-k}$

Solution:

To show that $\binom{n}{k} = \binom{n}{n-k}$ we can just use the definition of the binomial coefficient. On the left hand side we have $\binom{n}{k} = \dfrac{n!}{(n-k)! \times k!}$ and on the right hand side we have $\binom{n}{n-k} = \dfrac{n!}{(n-(n-k))! \times (n-k)!} = \dfrac{n!}{(n-n+k))! \times (n-k)!} = \dfrac{n!}{k! \times (n-k)!}$.

Through this algebraic manipulation we've shown that the right hand side and left hand side are the same.

With property 1 we can conclude that picking 2 balls from a box of 9 balls is the same as picking 7 balls from a box of 9 balls, because another way to think of this is picking which 2 balls to remove from 9 should be the same as picking which 7 to keep. We can also now know that when we want to determine the number of paths of length $a+b$ which we know is given by $|\mathcal{L}(a,b)| = \binom{a+b}{b}$, $\binom{a+b}{b} = \binom{a+b}{a}$.

Exercise 5.1

Using Property 1 of the Binomial Coefficient, determine what each of the following coefficients are equal to.

1. $\binom{6}{2}$
2. $\binom{9}{3}$
3. $\binom{6}{4}$
4. $\binom{14}{6}$
5. $\binom{128}{93}$

Exercise 5.1 Solution

1. $\binom{6}{2} = \binom{6}{4}$
2. $\binom{9}{3} = \binom{9}{6}$
3. $\binom{6}{4} = \binom{6}{2}$
4. $\binom{14}{6} = \binom{14}{8}$
5. $\binom{128}{93} = \binom{128}{35}$

Property 2 of the Binomial Coefficient

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

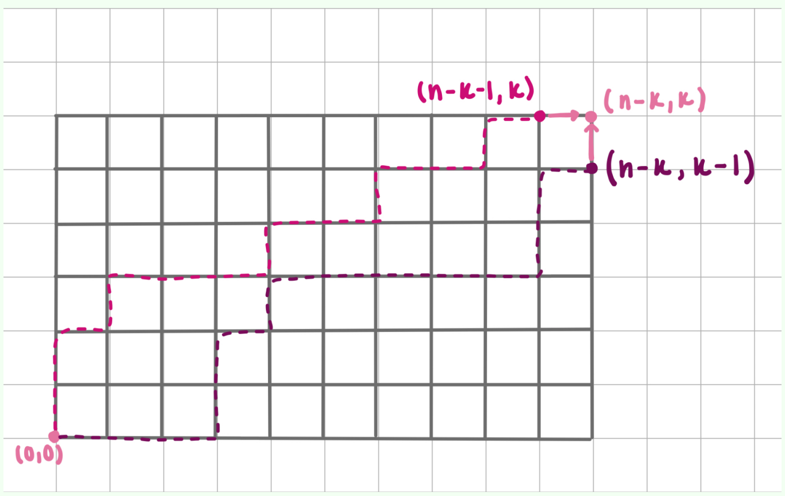
Example 5.2

Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ using lattice paths

Solution:

Recall that the number of paths of length $a+b$ which we know is given by $|\mathcal{L}(a,b)| = \binom{a+b}{b}$.

So $\binom{n}{k}$ is the number of paths of length $n-k+k$ and is given by $|\mathcal{L}(n-k,k)|$. $\binom{n-1}{k}$ is the number of paths of length $n-1-k+k$ and is given by $|\mathcal{L}(n-k-1,k)|$. And finally $\binom{n-1}{k-1}$ is the number of paths of length $n-1-(k-1)+(k-1)$ and is given by $|\mathcal{L}(n-k,k-1)|$. Lets use this to build a set of paths like below;



So we can see that the paths $\mathcal{L}(n-k-1,k)$ are all the paths of the set $\mathcal{L}(n-k,k)$ which we need to take one more step to the right to reach the point $(n-k,k)$ on our grid and similarly the paths $\mathcal{L}(n-k,k-1)$ are all the paths of the set $\mathcal{L}(n-k,k)$ which we need to take one more step up to reach the point $(n-k,k)$ on our grid. Therefore $|\mathcal{L}(n-k-1,k)|+|\mathcal{L}(n-k,k-1)| = |\mathcal{L}(n-k,k)|$ but this is exactly that $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$ as desired!